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ORIGINAL PAPER

Symmetry Solutions and Conserved Vectors of the Two-Dimensional Korteweg-de Vries Equation

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Abstract
We study the two-dimensional constant coefficients Korteweg-de Vries equation, which was established not long ago in the literature. We construct group-invariant solutions and conservation laws for this equation. Lie group method is applied and the Lie point symmetries are derived. We show how one can derive travelling waves symmetry solutions given in respect of the Weierstrass-zeta and hyperbolic functions using its symmetries. Furthermore, we present infinite number of conservation laws of the underlying equation obtained by means of the multiplier approach.

Keywords Two-dimensional Korteweg-de Vries equation · Lie point symmetries · Exact solution · Conservation laws · Multiplier method

Introduction

Physical phenomena in applied sciences and engineering is often best described by differential equations. These differential equations can be either the nonlinear ordinary differential equations (NODEs) or nonlinear partial differential equation (NPDEs) and these include amongst others the Schrödinger equation which plays significant role in quantum mechanics since it is the counterpart of the Newton's second law in the study of classical mechanics [1], the Boussinesq equation which is the model that describes the propagations of long waves in shallow water and in addition used in plasma physics as ion sound waves [2], the Kadomtsev-Petviashvili equation that is used as the model that describes the nonlin-

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ear propagation of electromagnetic pulses [3], the shallow water wave equation which is mostly used in fluid mechanics as the model that describes flow of fluids below a pressure surface [4], the equal-with equation which is used in simulating the wave propagation on nonlinear dispersion process medium [5], just to mention a few. It is vital to understand such occurrences from a mathematical position. Determining exact solutions of such equations is important in understanding the physical behaviour of the problems. Many solution methods have been introduced that could be employed to determine exact explicit solutions of differential equations. These methods are but not limited to the variable separated ODE technique [6], bifurcation technique [7], inverse scattering transform technique [8], Weierstrass elliptic function expansion technique [9], exponential function technique [10, 11], F-expansion method [12], Darboux transformation technique [13], (G'/G)-expansion technique [14, 15], tanh function technique [16–18], sine-cosine technique [19], homogeneous balance method [20], and symmetry analysis method [21].

The symmetry analysis technique introduced by Sophus Lie is a powerful method formed on the theory of Lie groups. Lie symmetry analysis is an efficient technique to analytically solve differential equations. This method decreases the amount of independent variables in the original system of partial differential equations (PDEs) and this results in a reduced system of differential equations. For a system of ordinary differential equations it will reduce the order of the system which makes it easier to find group-invariant solutions of a reduced system than the original system. For details on this method, see for example [21–30].

The renowned Korteweg-de Vries (KdV) equation [31] given by

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

is the mathematical model which describes long waves in shallow water surfaces. Here t and x denote time and position, respectively and $u(x, t)$ represents the wave surface. There have been several extensions of the KdV Eq. (1), namely the integrable constant and time dependent coefficients two and three dimensional KdV equations [32]. It was found that each equation was integrable by employing the Painlevè test and furthermore, Hirota's technique was invoked to compute multiple solitary wave solutions [32].

The extended constant coefficients two-dimensional KdV equation [32] reads

$$u_{ty} + \beta u_{xx} + \alpha \left(u_x u_{xy} + u_{xx} u_y\right) + u_{xxxy} + \gamma u_{yy} = 0, \tag{2}$$

with α, β and γ being constants. The above equation is an extension of that constant coefficients $(2+1)$-dimensional KdV equation, obtained by adding two terms [32].

In the present work, we investigate the two-dimensional KdV constant coefficients Eq. (2). Moreover, we find travelling waves group-invariant solutions by reducing (2). We then use multipliers to derive conserved vectors of the Eq. (2).

Exact Solutions of Eq. (2)

Lie Point Symmetries

The symmetry group of (2) will be generated by

$$\Gamma = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u},$$

with ξ^1, ξ^2, ξ^3 and η being depend on variables t, x, y, u. To find symmetries of (2), we first apply the fourth extension of Γ to (2) and get an overdetermined linear system comprising of nineteen partial differential equations which are homogeneous. These equations are

$$\begin{aligned}
&\xi^1_u = 0,\ \xi^1_y = 0,\ \xi^1_x = 0,\ \xi^1_{tx} = 0,\ \xi^2_y = 0,\ \xi^2_u = 0,\ \xi^3_u = 0,\ \xi^3_x = 0,\ \eta_{yu} = 0,\ \eta_{uu} = 0,\\
&3\eta_{xu} - \xi^1_{tx} = 0,\ \xi^1_t + 3\eta_u = 0,\ 3\xi^2_x - \xi^1_t = 0,\ \gamma\xi^1_t - \gamma\xi^3_y - \xi^3_t = 0,\\
&2\xi^1_{txx} + 3\alpha\eta_x - 3\xi^2_t = 0,\ 2\beta\eta_{xu} - \beta\xi^2_{xx} + \alpha\eta_{xy} = 0,\\
&\eta_{ty} + \beta\eta_{xx} + \eta_{xxxy} + \gamma\eta_{yy} = 0,\ \beta(\xi^1_t - 2\xi^2_x + \xi^3_y) + \alpha\eta_y = 0,\\
&3\eta_{tu} + \xi^1_{txxx} - 3\gamma\xi^1_{ty} + 3\alpha\eta_{xx} = 0.
\end{aligned}$$

The solution of above system gives the infinitesimals

$$\begin{aligned}
&\xi^1 = 6\alpha C_1,\ \ \xi^2 = \alpha\mathcal{F}_1(t),\ \ \xi^3 = \alpha\mathcal{F}_2(y-\gamma t),\\
&\eta = \mathcal{F}_3(t) + x\mathcal{F}_1'(t) - \beta\mathcal{F}_2(y-\gamma t),
\end{aligned}$$

with $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ being arbitrary functions of their arguments. To acquire physically interesting and significant solutions of Eq. (2), we could take $\mathcal{F}_1(t) = C_2 t + C_3,\ \mathcal{F}_2(y-\gamma t) = C_4(y-\gamma t) + C_5,\ \mathcal{F}_3(t) = C_6 t + C_7$. Thus, the infinitesimals now become

$$\begin{aligned}
&\xi^1 = 6\alpha C_1,\ \ \xi^2 = \alpha C_2 t + \alpha C_3,\ \ \xi^3 = \alpha C_4(y-\gamma t) + \alpha C_5,\\
&\eta = C_2 x - \beta C_4(y-\gamma t) - \beta C_5 + C_6 t + C_7
\end{aligned}$$

and consequently the Lie algebra of symmetries of Eq. (2) is spanned by seven vector fields, namely

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial t},\ \ \Gamma_2 = \frac{\partial}{\partial x},\ \ \Gamma_3 = \frac{\partial}{\partial y},\ \ \Gamma_4 = \frac{\partial}{\partial u},\ \ \Gamma_5 = t\frac{\partial}{\partial u},\\
&\Gamma_6 = \alpha t\frac{\partial}{\partial x} + x\frac{\partial}{\partial u},\ \ \Gamma_7 = \alpha(y-\gamma t)\frac{\partial}{\partial y} - \beta(y-\gamma t)\frac{\partial}{\partial u}.
\end{aligned}\tag{3}$$

We now obtain the group transformations generated by the above symmetries. For this purpose, we solve the following Lie equations with the initial conditions for each of the symmetries (3):

$$\begin{aligned}
&\frac{d\bar{t}}{da} = \xi^1(\bar{t},\bar{x},\bar{y},\bar{u}),\ \ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \xi^2(\bar{t},\bar{x},\bar{y},\bar{u})\ \ \bar{x}|_{a=0} = x,\\
&\frac{d\bar{y}}{da} = \xi^3(\bar{t},\bar{x},\bar{y},\bar{u}),\ \ \bar{y}|_{a=0} = y,\ \ \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{y},\bar{u}),\ \ \bar{u}|_{a=0} = u.
\end{aligned}\tag{4}$$

We obtain

$$\begin{aligned}
&T_1:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t+a_1, x, y, u),\\
&T_2:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x+a_2, y, u),\\
&T_3:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x, y+a_3, u),\\
&T_4:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x, y, u+a_4),\\
&T_5:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x, y, u+a_5 t),\\
&T_6:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x+\alpha a_6 t, y, u+a_6 x),\\
&T_7:\ (\bar{t},\bar{x},\bar{y},\bar{u}) \longrightarrow (t, x, e^{\alpha a_7}(y-\gamma t)+\gamma t, u-\beta(y-\gamma t)a_7).
\end{aligned}$$

Using the above group transformations we can now state the following theorem:

Theorem *If $u = \mathcal{H}(t, x, y)$ is the known solution for the KdV Eq. (2), then the following are also solutions corresponding to each of the transformations $T_1, ..., T_7$:*

$$\begin{aligned}
u_1 &= \mathcal{H}(t + a_1, x, y),\\
u_2 &= \mathcal{H}(t, x + a_2, y),\\
u_3 &= \mathcal{H}(t, x, y + a_3),\\
u_4 &= \mathcal{H}(t, x, y) + a_4,\\
u_5 &= \mathcal{H}(t, x, y) + a_5 t,\\
u_6 &= \mathcal{H}(t, x + \alpha a_6 t, y) + a_6 x,\\
u_7 &= \mathcal{H}(t, x, e^{\alpha a_7}(y - \gamma t) + \gamma t) - \beta(y - \gamma t)a_7.
\end{aligned}$$

Symmetry Reductions and Solutions

Reduction for the symmetries $\Gamma_1,\ \Gamma_2,\ \Gamma_3$

We begin by using the combination of the three translation symmetries, i.e., $\Gamma_1 + a\Gamma_2 + b\Gamma_3$, with a, b constants, whose characteristic equations yields the invariants $r = x - at$, $s = y - bt$ and $U = u$. This then transforms Eq. (2) to the NPDE

$$U_{rrrs} + \alpha U_r U_{rs} + \alpha U_{rr} U_s + \beta U_{rr} + \gamma U_{ss} - aU_{rs} - bU_{ss} = 0. \tag{5}$$

The symmetries of (5) are

$$R_1 = \frac{\partial}{\partial U},\quad R_2 = \frac{\partial}{\partial s},\quad R_3 = \frac{\partial}{\partial r},\quad R_4 = r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + \left(\frac{2ar}{\alpha} - \frac{4\beta s}{\alpha} - U\right)\frac{\partial}{\partial U}.$$

As above, when using the symmetry $R = cR_2 + R_3$, with c being a constant, we obtain the invariants $p = s - cr$, $f = U$, which reduces (5) to the NODE

$$f'''' - \frac{2\alpha}{c} f' f'' + \left(\frac{b}{c^3} - \frac{\beta}{c} - \frac{a}{c^2} - \frac{\gamma}{c^3}\right) f'' = 0,\ c \neq 0. \tag{6}$$

Equation (6) may be rewritten in the form

$$f'''' - Af'f'' - Bf'' = 0, \tag{7}$$

where $A = 2\alpha/c$ and $B = (\beta c^2 + ac + \gamma - b)/c^3$. Integration of Eq. (7) in reference to the variable p results in

$$f''' - \frac{A}{2} f'^2 - Bf' = k_1,\ k_1 = \text{constant}. \tag{8}$$

The Eq. (8) is invariant under the symmetry group generators $X = \partial/\partial p$ and $Y = \partial/\partial f$. The invariants of $Y^{[1]} = \partial/\partial f$ are $T(p, f) = p$ and $V(p, f, f') = f'$. By Lie's theorem the third invariant is $dV/dT = f''$. Thus, $f''' = d^2V/dT^2$ and hence the Eq. (8) is written as

$$V'' - \frac{A}{2} V^2 - BV = k_1,\quad ' \equiv \frac{d}{dT}. \tag{9}$$

Now the group generator $X = \partial/\partial p$ gets transformed to $\tilde{X} = \partial/\partial T$, which is a Lie point symmetry of the reduced Eq. (9). The invariants of $\tilde{X}^{[1]}$ are $W(T, V) = V$ and

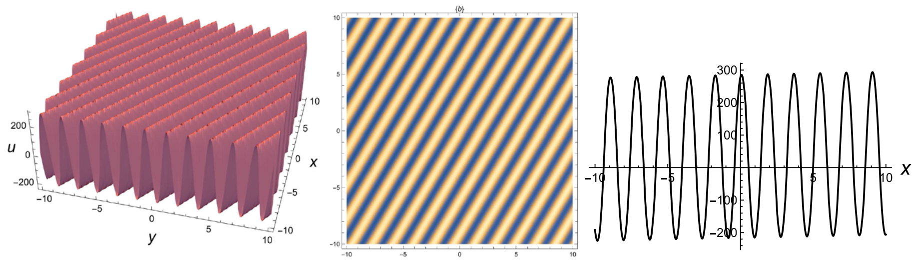

Fig. 1 The 3D and 2D profiles of the periodic solution (13)

$Z = (T, V, V') = V'$. Thus, we have $dZ/dW = V''/V'$. Hence the Eq. (9) becomes a first-order ODE

$$Z\frac{dZ}{dW} - \frac{A}{2}W^2 - BW = k_1. \tag{10}$$

Integration of the Eq. (10) and using the invariants W and Z gives

$$\frac{dV}{dT} = \pm\sqrt{\frac{A}{3}V^3 + BV^2 + 2k_1V + 2k_2}\,,\ k_2 = \text{constant}. \tag{11}$$

The general solution of (11) is given by

$$V(T) = \alpha_2 + (\alpha_1 - \alpha_2)\text{cn}^2\left\{\sqrt{\frac{A(\alpha_1 - \alpha_3)}{12}}T, R^2\right\},\ R^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \tag{12}$$

where cn is an elliptic cosine function, α_1, α_2 and α_3 are real roots of

$$V^3 - \frac{3B}{A}V^2 - \frac{6k_1}{A}V - \frac{6k_2}{A} = 0$$

with $\alpha_1 > \alpha_2 > \alpha_3$. Integration of (12) gives the following group-invariant solution of (2):

$$u(t,x,y) = \sqrt{\frac{12\,(\alpha_1 - \alpha_2)^2}{A(\alpha_1 - \alpha_3)R^8}}\left\{\text{EllipticE}\left(\text{sn}\left(\sqrt{\frac{A(\alpha_1 - \alpha_3)}{12}}p, R^2\right), R^2\right)\right\} + \left\{\alpha_2 - (\alpha_1 - \alpha_2)\,\frac{1 - R^4}{R^4}\right\}p + C_1, \tag{13}$$

where $p = (ac - b)t - cx + y$, $A = 2\alpha/c$, $B = (\beta c^2 + ac + \gamma - b)/c^3$, C_1 is a constant of integration, sn is an elliptic sine function and EllipticE$[q, \kappa]$ is an incomplete elliptic integral [33]

$$\text{EllipticE}[q, \kappa] = \int_0^q \sqrt{\frac{1 - \kappa^2 w^2}{1 - w^2}}dw.$$

The solution (13) is illustrated via the periodic graphs in Fig. 1 with the parametric values $A = 0.9,\ a = -4,\ b = 0.2,\ c = 0.6,\ \alpha_1 = 99,\ \alpha_2 = 52,\ \alpha_3 = -63,\ C_1 = 0.9$.

Some special solutions

(i) Suppose $\gamma = b$ and the coefficient $b/c^3 - \beta/c - a/c^2 - \gamma/c^3 = 0$, then NODE (6) becomes

$$f'''' + \frac{2\alpha\beta}{a}f'f'' = 0, \tag{14}$$

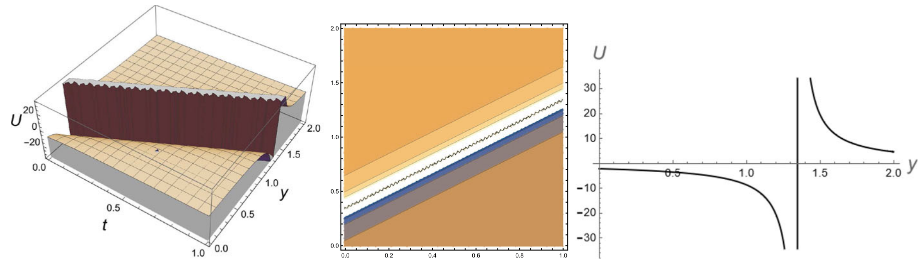

Fig. 2 The graphical demonstration of the solution (16)

and its solution is

$$f(p)=c_4-\sqrt[3]{\left(\frac{-6a}{\alpha\beta}\right)^2}\zeta\left(\sqrt[3]{\frac{-\alpha\beta}{6a}}(p+c_2);\,-2c_1\sqrt[3]{\frac{-6a}{\alpha\beta}},c_3\right), \tag{15}$$

where $\zeta(v;g_1,g_2)$ is the Weierstrass-zeta function and c_i's being constants. Thus, the group-invariant solution of KdV Eq. (2) for this case is

$$u(t,x,y)=-\sqrt[3]{\left(\frac{-6a}{\alpha\beta}\right)^2}\zeta\left(\sqrt[3]{\frac{-\alpha\beta}{6a}}\left\{\frac{a}{\beta}x+y-\left(b+\frac{a^2}{\beta}\right)t+c_2\right\};\,-2c_1\sqrt[3]{\frac{-6a}{\alpha\beta}},c_3\right)+c_4. \tag{16}$$

The solution (16) is traced graphically in Fig. 2, for certain desired values of arbitrary constants. Suitable values for the constants are taken as $a=2,\ b=-1,\ \alpha=-2,\ \beta=2,\ c_1=-0.35,\ c_2=-0.4,\ c_3=2,\ c_4=0$ at $x=0$.

(ii) We choose the constants (k_1, k_2) in Eq. (11) to be equal to zero. Then solving the resultant ODE leads to the solution

$$V=-\frac{3B}{A}\operatorname{sech}^2\left\{\frac{\sqrt{B}}{2}(T-k_3)\right\},\ k_3=\text{constant}. \tag{17}$$

Integrating the above value of V gives the solution f in the form

$$f(p)=-\frac{6\sqrt{B}}{A}\tanh\left\{\frac{\sqrt{B}}{2}(p-k_3)\right\}+k_4,\ k_4=\text{constant}. \tag{18}$$

Hence we get the hyperbolic solution to KdV Eq. (2) as

$$u(t,x,y)=K_0\tanh\left\{K_1\left((ac-b)t-cx+y+k_3\right)\right\}+k_4, \tag{19}$$

where the constants $K_0,\ K_1$ are expressed as

$$K_0=-\frac{3\sqrt{\beta c^2+ac+\gamma-b}}{\alpha c^{1/2}},\ \ K_1=\frac{\sqrt{\beta c^2+ac+\gamma-b}}{2c^{3/2}}.$$

Figure 3 gives the illustration of kink wave profile for the solution (19) along the $x,\ y$-axis at $t=5,\ a=0.1,\ b=0.1,\ c=1.5,\ K_0=1.7,\ K_1=5.6,\ k_3=0,\ k_4=11$.

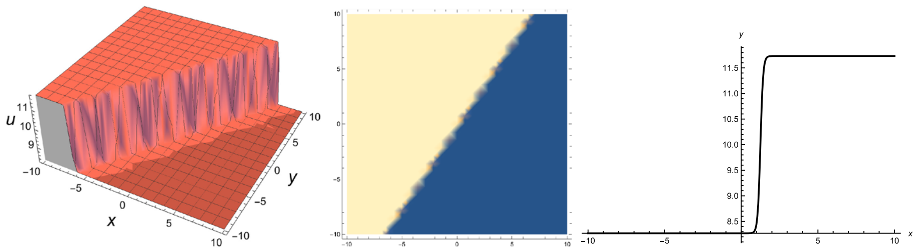

Fig. 3 Illustration of the kink wave solution (19)

Symmetry reduction for the symmetry Γ_6
This symmetry gives the invariants

$$f = t, \ g = y, \ U = u - \frac{x^2}{2\alpha t} \tag{20}$$

and the group-invariant solution for (2) is

$$u(t, x, y) = U(f, g) + \frac{x^2}{2\alpha t}. \tag{21}$$

Inserting the above value of u in KdV Eq. (2) gives the reduced NPDE

$$\alpha t U_{ty} + \beta + \alpha U_y + \alpha\gamma t U_{yy} = 0. \tag{22}$$

To solve Eq. (22), we let $z = \partial U/\partial y$ and this transforms Eq. (22) to the quasi-linear first-order PDE

$$\alpha t \frac{\partial z}{\partial t} + \alpha\gamma t \frac{\partial z}{\partial y} = -\frac{\beta + \alpha z}{\alpha}, \tag{23}$$

whose solution is

$$z(t, y) = \frac{1}{\alpha}\left\{\frac{1}{t}G(y - \gamma t) - \beta\right\}, \tag{24}$$

with G being arbitrary function of its argument. Consequently, we have

$$u(t, x, y) = \int \left\{\frac{1}{\alpha t}G(y - \gamma t)\right\} dy - \frac{\beta}{\alpha}y + \frac{x^2}{2\alpha t} + c_1, \tag{25}$$

with c_1 being arbitrary constant, as the group-invariant solution to KdV Eq. (2) under the symmetry Γ_6.

Figure 4 gives the periodic profile for the solution (25) with the arbitrary function G restricted to $G = \cos(y - \gamma t)$. Here we have chosen the parametric values to be $\alpha = 1, \ \beta = 0.1, \ \gamma = 1.8, \ c_1 = 1.6$ and $x = 0$.

Symmetry reduction for the symmetry Γ_7

We now consider the symmetry Γ_7 and use it to perform symmetry reduction. Solving the associated characteristic equations yield the invariants

$$j_1 = t, \ j_2 = x, \ \Phi = u + \frac{\beta}{\alpha}y. \tag{26}$$

The above provides group-invariant solution

$$u(t, x, y) = \Phi(t, x) - \beta y/\alpha. \tag{27}$$

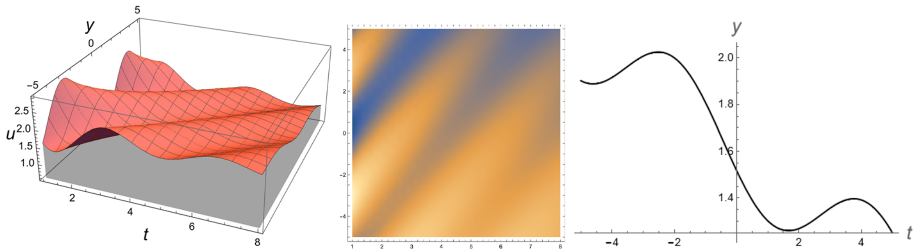

Fig. 4 Solution profiles of (25) with $G = \cos(y - \gamma t)$

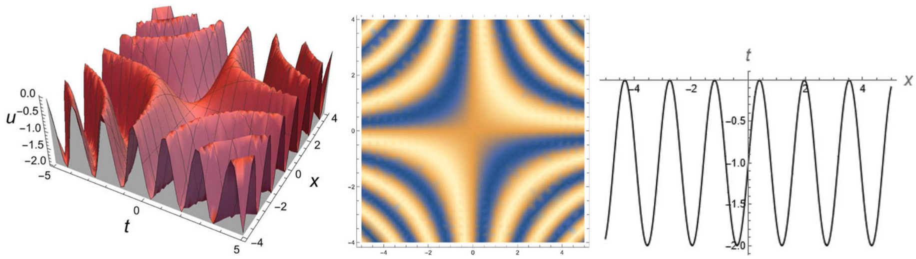

Fig. 5 Solution profiles for (28) with $\Phi = \sin(tx)$

The insertion of the value of u from (27) into (2), we see that this satisfies (2) for arbitrary Φ. Hence the group-invariant solution to KdV Eq. (2) for the symmetry Γ_7 is

$$u(t, x, y) = \Phi(t, x) - \beta y/\alpha, \tag{28}$$

where Φ is arbitrary.

Figure 5 demonstrates the periodic wave solution (28) for $\Phi = \sin(tx)$, $\alpha = 1$, $\beta = 0.1$ and $y = 10$.

Conservation Laws

We construct conservation laws of KdV Eq. (2) in this section. A conservation law in physics means that a certain property of a system does not change over time. For example, conservation of electric charge, energy, momentum and many others. Conservation laws are central to understanding the physical world, in that they describe which physical processes are possible. Conservation laws are vital in solving and reducing the order of differential equations. They are utilized to obtain exact solutions and numerical integration of PDEs and are of key importance in the study of the phenomena exhibited by them [34–37]. There are different methods that could be used to derive conservation laws of PDEs.

Here the multiplier method [27] will be utilized as the equation does not have a Lagrangian. Before we derive conservation laws for (2), we first recall some definitions and basic results of the multiplier method to be applied in this section.

For the two-dimensional KdV Eq. (2), a local continuity equation is a divergence expression

$$D_t T + D_x X + D_y Y|_{(2)} = 0, \tag{29}$$

where T is conserved density, X, Y are spatial fluxes and D_i is the total derivative [28]. Here T, X, Y depend on t, x, y, u.

Any non-trivial local conservation law (29) is analogous to

$$D_t\tilde{T} + D_x\tilde{X} + D_y\tilde{Y} = \left(u_{ty} + \beta u_{xx} + \alpha u_x u_{xy} + \alpha u_{xx}u_y + u_{xxxy} + \gamma u_{yy}\right) Q \quad (30)$$

vanishing on the solution space of (2), $Q(t, x, y, u)$ being the multiplier, and $(\tilde{T}, \tilde{X}, \tilde{Y})$ varies with (T, X, Y) by a trivial conserved current. On solution space of Eq. (2), the form (30) reduces to (29).

We will consider multipliers of zeroth order, that is $Q = Q(t, x, y, u)$. The determining equation for obtaining all multipliers Q is given by

$$\frac{\delta}{\delta u}Q(u_{ty} + \beta u_{xx} + \alpha u_x u_{xy} + \alpha u_{xx}u_y + u_{xxxy} + \gamma u_{yy}) = 0. \quad (31)$$

Here $\delta/\delta u$ is Euler operator which is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s\geq 1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u_{i_1 i_2 \dots i_s}}. \quad (32)$$

After expanding the determining Eq. (31) and solving for Q, we obtain

$$Q = f_1(t) + f_2(y - \gamma t)$$

for arbitrary functions f_1 and f_2. We now employ the homotopy formula [27] to find the conserved vectors. The conserved vectors for (2) that corresponds to the two multipliers $f_1(t)$ and $f_2(y - \gamma t)$ are given, respectively, by

$$\begin{aligned}
T^t_{f_1} &= \frac{1}{2}u_y f_1(t),\\
T^x_{f_1} &= \left(\beta u_x + \frac{3}{4}u_{xxy} - \frac{1}{4}\alpha u u_{xy} + \frac{3}{4}\alpha u_x u_y\right) f_1(t),\\
T^y_{f_1} &= \left(\frac{1}{2}u_t + \gamma u_y + \frac{1}{4}\alpha u_x^2 + \frac{1}{4}\alpha u u_{xx} + \frac{1}{4}u_{xxx}\right) f_1(t) - \frac{1}{2}u f_1{}'(t);\\
T^t_{f_2} &= \frac{1}{2}u_y f_2(y-\gamma t) - \frac{1}{2}u f_2{}'(y-\gamma t),\\
T^x_{f_2} &= \left(\frac{3}{4}\alpha u_x u_y + \beta u_x + \frac{3}{4}u_{xxy} - \frac{1}{4}\alpha u u_{xy}\right) f_2(y-\gamma t)\\
&\quad - \left(\frac{1}{4}\alpha u u_x + \frac{1}{4}u_{xx}\right) f_2{}'(y-\gamma t),\\
T^y_{f_2} &= \left(\gamma u_y + \frac{1}{4}\alpha u_x^2 + \frac{1}{4}\alpha u u_{xx} + \frac{1}{4}u_{xxx} + \frac{1}{2}u_t\right) f_2(y-\gamma t) - \frac{1}{2}\gamma u f_2{}'(y-\gamma t).
\end{aligned}$$

Remark Since the multiplier has arbitrary functions f_1 and f_2, it means that there are infinitely many nonlocal conservation laws for the KdV Eq. (2).

Concluding Remarks

In this paper, we constructed the group invariant solutions and an infinite number of conserved vectors for the new two-dimensional KdV equation with constant coefficients by means of Lie group symmetry method and the multiplier method, respectively. The Lie algebra spanned by Lie point symmetries admitted by the equation was derived. The obtained exact solutions are invariant under the subalgebra of combination of temporal and spatial translation symmetries of equation. These subalgebras gave rise to the special group-invariant solutions known as travelling wave solutions that have a constant velocity throughout their course of propagation [27]. Here in this work we succeeded in obtaining such physically important solutions presented in terms of the incomplete elliptic integral, Weierstrass-zeta and hyperbolic functions which are valuable to study new phenomena emerge in the novel equation we investigated. Moreover, we have constructed an infinite number of conserved vectors of the equation which manifests on the integrability of underlying equation. The obtained results we presented here due to the applications of the methods we chose are new and have not been divulged in the literature earlier.

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Declarations

Conflict of interest/ Competing interests The authors declare that they have no conflict of interest/competing interests.

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